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FIXED POINT THEOREM FOR FOUR METRIC SPACES

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ABSTRACT

The aim of this paper is to obtain some fixed point theorems for four metric spaces which is a generalization of the results of Jain, Sahu and Fisher[2] and the result of Nung[1] for three metric spaces.

Keywords: Cauchy Sequences, complete metric space, fixed points.

I. INTRODUCTION

Definition 1.1 - Let X be a non empty set and a mapping $d: X \times X \rightarrow \mathbf{R}$ is satisfies following conditions for all x, y, z $\in$ X:

(M_1) $d(x, y) \geq 0$, (non negativity),
(M_2) $d(x, y) = 0$ if and only if $x = y$, (identity),
(M_3) $d(x, y) = d(y, x)$, (symmetry),
(M_4) $d(x, y) \leq d(x, z) + d(z, y)$,(triangle inequality).

Then d is said to be *metric* on X, or in other words pair (X, d) is called *metric space*.

Definition 1.2 - Let $\{x_n\}$ be a sequence in metric space (X, d) is said to be:
(i) *Converge* to x, if any $\varepsilon > 0$, $d(x_n, x) < \varepsilon$ for all $n \geq m$, there exist $m \in \mathbf{N}$ depending upon ε.
(ii) *Cauchy sequence*, if any $\varepsilon > 0$, $d(x_{n+p}, x_n) < \varepsilon$, $p > 0$, $n \geq m$, there exist $m \in \mathbf{N}$ depending upon ε.

Definition 1.3 - A metric space (X, d) is said to be *complete* if and only if every Cauchy sequence in X is convergent.

The following fixed point theorems were proved by Jain, Sahu and Fisher[2] and Nung[1].

Theorem A[2] Let (X, d), (Y, ρ) and (Z, σ) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X satisfying the inequalities:
(i) $d(RSTx, RSTx') \leq c \max\{d(x, x'), d(x, RSTx), d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}$,
(ii) $\rho(TRSy, TRSy') \leq c \max\{\rho(y, y'), \rho(y, TRSy), \rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}$,
(iii) $\sigma(STRz, STRz') \leq c \max\{\sigma(z, z'), \sigma(z, STRz), \sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}$,

for all $x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$ where $0 \leq c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further $Tu = v$, $Sv = w$ and $Rw = u$.

Theorem B[1] Let (X, d), (Y, ρ) and (Z, σ) be complete metric spaces. If T is a continuous mapping of X into Y, S is a continuous mapping of Y into Z and R is a continuous mapping of Z into X satisfying the inequalities
(i) $d(RSTx, RSy) \leq c \max\{d(x, RSy), d(x, RSTx), \rho(y, Tx), \sigma(Sy, STx)\}$,
(ii) $\rho(TRSy, TRz) \leq c \max\{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), d(Rz, RSy)\}$,
(iii) $\sigma(STRz, STx) \leq c \max\{\sigma(z, STx), \sigma(z, STRz), d(x, Rz), \rho(Tx, TRz)\}$,

for all $x \in X$, $y \in Y$ and $z \in Z$ where $0 \leq c < 1$, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

II. MAIN RESULTS

We want to prove the following results which are related to fixed point theorems of Theorem A and Theorem B respectively.

Theorem 2.1 - Let (X, d), (Y, ρ), (Z, σ) and (W, δ) be four complete metric spaces. If four continuous mappings $T: X \to Y$, $R: Y \to Z$, $S: Z \to W$ and $U: W \to X$ satisfying the inequalities:

(i) $d(USRTx, USRTx') \leq c \max\{d(x,x'), d(x, USRTx), d(x', USRTx'), \rho(Tx, Tx'), \sigma(RTx, RTx'), \delta(SRTx, SRTx')\}$,

(ii) $\rho(TUSRy, TUSRy') \leq c \max\{\rho(y, y'), \rho(y, TUSRy), \rho(y', TUSRy'), \sigma(Ry, Ry'), \delta(SRy, SRy'), d(USRy, USRy')\}$,

(iii) $\sigma(RTUSz, RTUSz') \leq c \max\{\sigma(z, z'), \sigma(z, RTUSz), \sigma(z', RTUSz'), \delta(Sz, Sz'), d(USz, USz'), \rho(TUSz, TUSz')\}$,

(iv) $\delta(SRTUw, SRTUw') \leq c \max\{\delta(w, w'), \delta(w, SRTUw), \delta(w', SRTUw'), d(Uw, Uw'), \rho(TUw, TUw'), \sigma((RTUw, RTUw')\}$.

for all x, x' ϵ X, y, y' ϵ Y, z, z' ϵ Z and w, w' ϵ W, where $0 \leq c < 1$. Then USRT, TUSR, RTUS and SRTU has unique fixed points h, k, p and q in X, Y, Z and W respectively. Further, Th = k, Rk = p, Sp = q and Uq = h.

Proof – Let x_0 be an arbitrary point in X and suppose that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X, Y, Z and W respectively by

$$x_n = (USRT)^n x_0,\ y_n = Tx_{n-1},\ z_n = Ry_n, \text{ and } w_n = Sz_n \text{ for } n = 1, 2, 3, \ldots \quad \textbf{(1)}$$

Now using the inequality (i), we observe that

(a)
$$\begin{aligned} d(x_n, x_{n+1}) &= d(USRTx_{n-1}, USRTx_n) \\ &\leq c \max\{d(x_{n-1}, x_n), d(x_{n-1}, USRTx_{n-1}), d(x_n, USRTx_n), \rho(Tx_{n-1}, Tx_n), \sigma(RTx_{n-1}, RTx_n), \delta(SRTx_{n-1}, SRTx_n)\} \\ &= c \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}), \delta(w_n, w_{n+1})\} \quad \textbf{[from (1)]} \\ &= c \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n)\}. \end{aligned}$$

Now applying inequality (ii), we conclude that

(b)
$$\begin{aligned} \rho(y_n, y_{n+1}) &= \rho(TUSR\, y_{n-1}, TUSR\, y_n) \\ &\leq c \max\{\rho(y_{n-1}, y_n), \rho(y_{n-1}, TUSRy_{n-1}), \rho(y_n, TUSRy_n), \sigma(Ry_{n-1}, Ry_n), \delta(SRy_{n-1}, SRy_n), d(USRy_{n-1}, USRy_n)\} \\ &= c \max\{\rho(y_{n-1}, y_n), \rho(y_{n-1}, y_n), \rho(y_n, y_{n+1}), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n)\} \quad \textbf{[from(1)]} \\ &= c \max\{\rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n)\} \end{aligned}$$

And similarly by (iii) and (iv), we can write

(c) $\sigma(z_n, z_{n+1}) \leq c \max\{\sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}$,

(d) $\delta(w_n, w_{n+1}) \leq c \max\{\delta(w_{n-1}, w_n), d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n)\}$

Now by induction the inequalities (a), (b), (c) and (d) are becomes as :

When $0 \leq c < 1$, we have

$$d(x_n, x_{n+1}) \leq c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \delta(w_1, w_2)\},$$

$$\rho(y_n, y_{n+1}) \leq c^{n-1} \max\{\rho(y_1, y_2), \sigma(z_1, z_2), \delta(w_1, w_2), d(x_1, x_2)\},$$

$$\sigma(z_n, z_{n+1}) \leq c^{n-1} \max\{\sigma(z_1, z_2), \delta(w_1, w_2), d(x_1, x_2), \rho(y_1, y_2)\},$$

$$\delta(w_n, w_{n+1}) \leq c^{n-1} \max\{\delta(w_1, w_2), d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}.$$

It follows that the sequences are $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences with limits h, k, p and q respectively. We know that T, R and S are continuous, therefore we get

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{n-1} = Th = k,$$
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Ry_n = Rk = p,$$
and
$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} Sz_n = Sp = q.$$

Now taking $n\to\infty$ and $0 \le c < 1$, we observe that

$$\begin{aligned} d(USRTh, x_n) &= d(USRTh, USRTx_{n-1}) \\ &\le c \max\{d(h, x_{n-1}), d(h, USRTh), d(x_{n-1}, USRTx_{n-1}), \rho(Th, Tx_{n-1}), \sigma(RTh, RTx_{n-1}), \delta(SRTh, SRTx_{n-1})\} \\ &= c \max\{d(h, h), d(h, USRTh), d(h, USRTh), \rho(k, k), \sigma(p, p), \delta(q, q)\} \\ &= c\, d(h, USRTh) \end{aligned}$$

Thus USRTh = h and h is fixed point of USRT.

In the same way it is obvious to show that

$$TUSR(k) = TUSR(Th) = T(USRTh) = Th = k,$$
$$RTUS(p) = RTUS(Rk) = R(TUSRk) = Rk = p,$$
$$SRTU(q) = SRTU(Sp) = S(RTUSp) = Sp = q.$$

Hence, k, p and q are fixed points of TUSR, RTUS and SRTU respectively.

Uniqueness of h

Suppose(hyp.) h' is a second fixed point of USRT and $h \neq h'$, so that

using (i)

$$\begin{aligned} d(h, h') &= d(USRTh, USRTh') \\ &\le c \max\{d(h, h'), d(h, USRTh), d(h', USRTh'), \rho(Th, Th'), \sigma(RTh, RTh'), \delta(SRTh, SRTh')\} \\ &= c \max\{\rho(Th, Th'), \sigma(RTh, RTh'), \delta(SRTh, SRTh')\}, \end{aligned}$$

using (ii)

$$\begin{aligned} \rho(Th, Th') &= \rho(TUSR\,Th, TUSR\,Th') \\ &\le c \max\{\rho(Th, Th'), \rho(Th, TUSRTh), \rho(Th', TUSRTh'), \sigma(RTh, RTh'), \delta(SRTh, SRTh'), d(USRTh, USRTh')\} \\ &= c \max\{\sigma(RTh, RTh'), \delta(SRTh, SRTh'), d(h, h')\} \\ &= c \max\{\sigma(RTh, RTh'), \delta(SRTh, SRTh')\} \end{aligned}$$

using (iii)

$$\begin{aligned} \sigma(RTh, RTh') &= \sigma(RTUS\,RTh, RTUS\,RTh') \\ &\le c \max\{\sigma(RTh, RTh'), \sigma(RTh, RTUSRTh), \sigma(RTh', RTUSRTh'), \delta(SRTh, SRTh'), d(USRTh, USRTh'), \rho(TUSRTh, TUSRTh')\}, \\ &= c \max\{\delta(SRTh, SRTh'), d(h, h'), \rho(Th, Th')\}, \\ &= c \max\{\delta(SRTh, SRTh'), d(h, h')\} \end{aligned}$$

Hence $d(h, h') \le c.\ \delta(SRTh, SRTh')$

using (iv)

$$\begin{aligned} d(h, h') &\le c.\ \delta(SRTh, SRTh') \\ &= c.\delta(SRTU(SRTh), SRTU(SRTh')) \\ &\le c.c \max\{\delta(SRTh, SRTh'), \delta(SRTh, SRTU\,SRTh), \delta(SRTh', SRTU\,SRTh'), d(USRTh, USRTh'), \rho(TUSRTh, TUSRTh'), \sigma(RTUSRTh, RTUSRTh')\}, \\ &= c^2 \max\{\delta(SRTh, SRTh'), d(h, h'), \rho(Th, Th'), \sigma(RTh, RTh')\}, \\ &= c^2\, d(h, h') \end{aligned}$$

Since $0 \le c < 1$, therefore h = h'.

Similarly we can show the uniqueness of k, p and q.

Now, we show Uq = h. Writing Uq = U(SRTUq) = USRT(Uq).

So, Uq is fixed point of USRT. But h is unique fixed point of USRT therefore Uq = h■

Theorem 2.2 - Let (X, d), (Y, ρ), (Z, σ) and (W, δ) be four complete metric spaces. If four continuous mappings T: $X\to Y$, R: $Y\to Z$, S: $Z\to W$ and U: $W\to X$ satisfying the inequalities:

(i) $d(USRTx, USRy) \le c \max\{d(x, USRTx), d(x, USRy), \rho(y, Tx), \sigma(Ry, RTx), \delta(SRy, SRTx)\}$,

(ii) $\rho(TUSRy, TUSz) \le c \max\{\rho(y, TUSRy), \rho(y, TUSz), \sigma(z, Ry), \delta(Sz, SRy), d(USz, USRy)\}$,

(iii) $\sigma(RTUSz, RTUw) \le c \max\{\sigma(z, RTUSz), \sigma(z, RTUw), \delta(w, Sz), d(Uw, USz), \rho(TUw, TUSz)\}$,

(iv) $\delta(SRTUw, SRTx) \le c \max\{\delta(w, SRTUw), \delta(w, SRTx), d(x, Uw), \rho(Tx, TUw), \sigma((RTx, RTUw)\}$,

for all $x \in X$, $y \in Y$, $z \in Z$ and $w \in W$ where $0 \le c < 1$. Then USRT, TUSR, RTUS and SRTU have unique fixed points h, k, p and q in X, Y, Z and W res-pectively. Further Th = k, Rk = p, Sp = q and Uq = h.

Proof – Let x_0 be an arbitrary point in X. Define a sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X, Y, Z and W respectively by

$$x_n = (USRT)^n x_0,\ y_n = Tx_{n-1},\ z_n = Ry_n, \text{ and } w_n = Sz_n \text{ for } n = 1, 2, 3, \ldots \quad \textbf{(1)}$$

By using inequality (i),

(a) $d(x_n, x_{n+1}) = d(x_{n+1}, x_n)$
$= d(USRTx_n, USRy_n)$
$\leq c \max\{d(x_n, USRTx_n), d(x_n, USRy_n), \rho(y_n, Tx_n), \sigma(Ry_n, RTx_n), \delta(SRy_n, SRTx_n)\}$
$= c \max\{d(x_n, x_{n+1}), d(x_n, x_n), \rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}), \delta(w_n, w_{n+1})\}$ **[from (1)]**
$= c \max\{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n)\}$.

Now applying inequality (ii),

(b) $\rho(y_n, y_{n+1}) = \rho(y_{n+1}, y_n)$
$= \rho(TUSRy_n, TUSz_{n-1})$
$\leq c \max\{\rho(y_n, TUSRy_n), \rho(y_n, TUSz_{n-1}), \sigma(z_{n-1}, Ry_n), \delta(Sz_{n-1}, SRy_n), d(USz_{n-1}, USRy_n)\}$
$= c \max\{\rho(y_n, y_{n+1}), \rho(y_n, y_n), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n)\}$ **[from(1)]**
$= c \max\{\rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n)\}$

And similarly by (iii) and (iv), we write

(c) $\sigma(z_n, z_{n+1}) \leq c \max\{\sigma(z_{n-1}, z_n), \delta(w_{n-1}, w_n), d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}$,

(d) $\delta(w_n, w_{n+1}) \leq c \max\{\delta(w_{n-1}, w_n), d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n)\}$

Now by induction the inequalities (a), (b), (c) and (d) are becomes as :

When $0 \leq c < 1$, we have

$d(x_n, x_{n+1}) \leq c^{n-1} \max\{d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2), \delta(w_1, w_2)\}$,
$\rho(y_n, y_{n+1}) \leq c^{n-1} \max\{\rho(y_1, y_2), \sigma(z_1, z_2), \delta(w_1, w_2), d(x_1, x_2)\}$,
$\sigma(z_n, z_{n+1}) \leq c^{n-1} \max\{\sigma(z_1, z_2), \delta(w_1, w_2), d(x_1, x_2), \rho(y_1, y_2)\}$,
$\delta(w_n, w_{n+1}) \leq c^{n-1} \max\{\delta(w_1, w_2), d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2)\}$.

It follows that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences with limits h, k, p and q respectively. We know that R and S are continuous, therefore we get

$\lim_{n\to\infty} x_n = h$,
$\lim_{n\to\infty} z_n = \lim_{n\to\infty} Ry_n = Rk = p$,
and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} Sz_n = Sp = q$.

Now taking $n\to\infty$ and $0 \leq c < 1$, we observe that

$d(x_{n+1}, USR\,k) = d(USRTx_n, USR\,k)$ **[from (1)]**
$\leq c \max\{d(x_n, USRT\,x_n), d(x_n, USR\,k), \rho(k, Tx_n), \sigma(Rk, RTx_n), \delta(SRk, SRTx_n)\}$

we have, $d(h, USR\,k) \leq 0$
So, $USRk = h$,
or $USp = h$,
or $Uq = h$

Now by **using(ii),**

$\rho(y_{n+1}, Th) = \rho(TUSR\,y_n, T\,USp)$
$= \rho(TUSRy_n, TUSp)$
$\leq c \max\{\rho(y_n, TUSRy_n), \rho(y_n, TUSp), \sigma(p, Ry_n), \delta(Sp, SRy_n), d(USp, USRy_n)\}$

When $n\to\infty$, we have
$\rho(k, Th) \leq 0$,
or $Th = k$

we now have,

$USRT(h) = USR(Th) = USRk = USp = Uq = h$,
$TUSR(k) = TUSR(Th) = T(USRTh) = Th = k$,
$RTUS(p) = RTUS(Rk) = R(TUSRk) = Rk = p$,
$SRTU(q) = SRTU(Sp) = S(RTUSp) = Sp = q$.

Hence, h, k, p and q are fixed point of USRT, TUSR, RTUS and SRTU respectively.

Uniqueness of h

Suppose(hyp.) h' is a second fixed point of USRT and $h \neq h'$, so that **using (i)**

$$
\begin{aligned}
d(h, h') &= d(USRTh, USRTh') \\
&\leq c \max\{d(h, USRTh), d(h, USRTh'), \rho(Th', Th), \sigma(RTh', RTh), \delta(SRTh', SRTh)\}, \\
&= c \max\{\rho(Th', Th), \sigma(RTh', RTh), \delta(SRTh', SRTh)\}, \text{ using (ii)}
\end{aligned}
$$

$$
\begin{aligned}
\rho(Th', Th) &= \rho(TUSR\,Th', TUSR\,Th) \\
&= \rho(TUSR\,Th', TUS\,RTh) \\
&\leq c \max\{\rho(Th', TUSRTh'), \rho(Th', TUSRTh), \sigma(RTh, RTh'), \delta(SRTh, SRTh'), d(USRTh, USRTh')\} \\
&= c \max\{\sigma(RTh, RTh'), \delta(SRTh, SRTh'), d(h, h')\}, \\
&= c \max\{\sigma(RTh, RTh'), \delta(SRTh, SRTh')\},
\end{aligned}
$$

using (iii)

$$
\begin{aligned}
\sigma(RTh, RTh') &= \sigma(RTUS\,RTh, RTUS\,RTh') \\
&= \sigma(RTUS\,RTh, RTU\,SRTh') \\
&\leq c \max\{\sigma(RTh, RTUSRTh), \sigma(RTh, RTU\,RTh'), \delta(SRTh', SRTh), d(USRTh', USRTh), \\
&\qquad \rho(TUSRTh', TUSRTh)\}, \\
&= c \max\{\delta(SRTh', SRTh), d(h', h), \rho(Th', Th)\}, \\
&= c \max\{\delta(SRTh', SRTh), d(h,, h)\},
\end{aligned}
$$

Hence $d(h, h') \leq c.\ \delta(SRTh, SRTh')$

using (iv)

$$
\begin{aligned}
d(h, h') &\leq c.\ \delta(SRTh, SRTh') \\
&= c.\delta(SRTU(SRTh), SRTU(SRTh')) \\
&= c.\delta(SRTU(SRTh), SRT(USRTh')) \\
&= c.\delta(SRTU(SRTh), SRT(h')) \\
&\leq c.\ c \max\{\delta(SRTh, SRTU\,SRTh), \delta(SRTh, SRTh'), d(h', USRTh), \rho(Th', TUSRTh), \sigma(RTh', RTUSRTh)\} \\
&= c^2 \max\{\delta(SRTh, SRTh'), d(h', h), \rho(Th', Th), \sigma(RTh', RTh)\}, \\
&= c^2\, d(h', h) \\
&= c^2\, d(h, h') \qquad \textbf{[from } M_3\textbf{]}
\end{aligned}
$$

Since $0 \leq c < 1$, therefore h = h'. Similarly we can show that the uniqueness of k, p and q■

REFERENCES

1. *R. K. Jain, H. K. Sahu and B. Fisher, "Related fixed point theorems for three metric spaces", Novi Sad J. Math. 26(1) (1996), pp. 11-17.*
2. *N. P. Nung, "A fixed point theorem in three metric spaces", Math. Sem. Notes, Kobe Univ. 11(1983), pp. 77-79.*